Bounded solutions of neutral fermions with a screened Coulomb potential

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Abstract

The intrinsically relativistic problem of a fermion subject to a pseudoscalar screened Coulomb plus a uniform background potential in two-dimensional space-time is mapped into a Sturm-Liouville. This mapping gives rise to an effective Morse-like potential and exact bounded solutions are found. It is shown that the uniform background potential determinates the number of bound-state solutions. The behaviour of the eigenenergies as well as of the upper and lower components of the Dirac spinor corresponding to bounded solutions is discussed in detail and some unusual results are revealed. An apparent paradox concerning the uncertainty principle is solved by recurring to the concepts of effective mass and effective Compton wavelength.

1 Introduction

The Coulomb potential of a point electric charge in a 1+1 dimension, considered as the time component of a Lorentz vector, is linear ($\sim |x|$) and so it provides a constant electric field always pointing to, or from, the point charge. This problem is related to the confinement of fermions in the Schwinger and in the massive Schwinger models [1]-[2] and in the Thirring-Schwinger model [3]. It is frustrating that, due to the tunneling effect (Klein´s paradox), there are no bound states for this kind of potential regardless of the strength of the potential [4]-[5]. The linear potential, considered as a Lorentz scalar, is also related to the quarkonium model in one-plus-one dimensions [6]-[7]. Recently it was incorrectly concluded that even in this case there is solely one bound state [8]. Later, the proper solutions for this last problem were found [9]-[11]. However, it is well known from the quarkonium phenomenology in the real 3+1 dimensional world that the best fit for meson spectroscopy is found for a convenient mixture of vector and scalar potentials put by hand in the equations (see, e.g., [12]). The same can be said about the treatment of the nuclear phenomena describing the influence of the nuclear medium on the nucleons [13]-[22]. The mixed vector-scalar potential has also been analyzed in 1+1 dimensions for a linear potential [23] as well as for a general potential which goes to infinity as $|x| \rightarrow \infty$ [24]. In both of those last references it has been concluded that there is confinement if the scalar coupling is of sufficient intensity compared to the vector coupling. Although the vector Coulomb potential does not hold relativistic bound-state solutions, its screened version ($\sim e^{-|x|/\lambda}$) is a genuine binding potential and its solutions have been found for fermions [25]. The screened Coulomb potential has also been analyzed with a scalar coupling in the Dirac equation [26] as well as in the Klein-Gordon equation with vector [27] and scalar [28] couplings.

The confinement of fermions by a pseudoscalar double-step potential [29] and their scattering by a pseudoscalar step potential [30] have already been analyzed in the literature providing the opportunity to find some quite interesting results. Indeed, the two-dimensional version of the anomalous magnetic-like interaction linear in the radial coordinate, christened by Moshinsky and Szczepaniak [31] as Dirac oscillator, has also received attention. Nogami and Toyama [32], Toyama et al. [33] and Toyama and Nogami [34] studied the behaviour of wave packets under the influence of that conserving-parity potential whereas Szmytkowski and Gruchowski [35] proved the completeness of the eigenfunctions. More recently Pacheco et al. [36] studied

some thermodynamics properties of the 1+1 dimensional Dirac oscillator, and a generalization of the Dirac oscillator for a negative coupling constant was presented in Ref. [37]. The two-dimensional generalized Dirac oscillator plus an inversely linear potential has also been addressed [38]. In recent papers, Villalba [39] and McKeon and Van Leeuwen [40] considered a pseudoscalar Coulomb potential ($V = \lambda/r$) in 3+1 dimensions and concluded that there are no bounded solutions. The reason attributed in Ref. [40] for the absence of bounded solutions it that the different parity eigenstates mix. Furthermore, the authors of Ref. [40] assert that *the absence of bound states in this system confuses the role of the π-meson in the binding of nucleons.* Such an intriguing conclusion sets the stage for the analyses by other sorts of pseudoscalar potentials. A natural question to ask is if the absence of bounded solutions by a pseudoscalar Coulomb potential is a characteristic feature of the four-dimensional world. In Ref. [37] the Dirac equation in one-plus-one dimensions with the pseudoscalar power-law potential $V = \mu|x|^\delta$ was approached and there it was concluded that only for $\delta > 0$ there can be a binding potential. Furthermore, we succeed in searching bound-state solutions for the pseudoscalar Coulomb potential ($V = m\omega|x|$) as well as for the pseudoscalar potential $V = m\omega x$, even in the case $\omega < 0$ [37].

In the present paper we begin our presentation of the Dirac equation with the most general potential in 1+1 dimensions. This general approach allows us to show that the pseudoscalar potential has a clearly different behaviour when compared with the scalar and vector potentials, even in the nonrelativistic limit of the theory. It was shown in Ref. [38] that the problem of a fermion under the influence of a pseudoscalar potential can always be mapped into a Sturm-Liouville problem for the upper and lower components of the Dirac spinor. Following that same methodology the present paper shows that the screened Coulomb potential plus a uniform background potential gives rise to an effective Morse-like potential, in the same way as the generalized Dirac oscillator plus an inversely linear potential gives rise to an effective quadratic plus inversely quadratic potential [38]. The exact bounded solutions are found, such as in Ref. [38], by transforming the Sturm-Liouville problem into Kummer's equation and expressing the upper and the lower components of the Dirac spinor in terms of confluent hypergeometric functions. In addition to their intrinsic importance as a new solution of the Dirac equation, this problem highlights the essential role of the uniform background potential in furnishing bounded solutions and it might be relevant to studies of binding of neutral fermions by electric fields. Furthermore, the

result renders a new contrast to the result found in Ref. [40]. In the nonrelativistic quantum mechanics, the problem of a particle subject to the Morse potential ($V = V_0 \left[1 - \exp(-ar)\right]^2$) has been used to describe the vibrations of nuclei in homonuclear diatomic molecules [41]-[42]. This sort of problem for S-wave states is transformed into the problem of solving a transcendent equation which is only approximately solvable [43]-[44]. Nevertheless, the one-dimensional asymmetric Morse potential ($-\infty < r < \infty$) is an exactly solvable problem in the nonrelativistic quantum mechanics [45]-[46], even if its parameters are complex numbers [47].

2 The Dirac equation in a 1+1 dimension

Thee 1+1 dimensional time-independent Dirac equation for a fermion of rest mass m reads

$$\mathcal{H}\Psi = E\Psi, \quad \mathcal{H} = c\alpha p + \beta mc^2 + \mathcal{V} \tag{1}$$

where E is the energy of the fermion, c is the velocity of light and p is the momentum operator. We use $\alpha = \sigma_1$ and $\beta = \sigma_3$, where σ_1 and σ_3 are Pauli matrices. For the potential matrix we consider

$$\mathcal{V} = 1V_t + \beta V_s + \alpha V_e + \beta\gamma^5 V_p \tag{2}$$

where 1 stands for the 2×2 identity matrix and $\beta\gamma^5 = \sigma_2$. This is the most general combination of Lorentz structures for the potential matrix because there are only four linearly independent 2×2 matrices. The subscripts for the terms of potential denote their properties under a Lorentz transformation: t and e for the time and space components of the 2-vector potential, s and p for the scalar and pseudoscalar terms, respectively. It is worth to note that the Dirac equation is covariant under $x \to -x$ if $V_e(x)$ and $V_p(x)$ change sign whereas $V_t(x)$ and $V_s(x)$ remain the same. This is because the parity operator $P = \exp(i\eta)P_0\sigma_3$, where η is a constant phase and P_0 changes x into $-x$, changes sign of α and $\beta\gamma^5$ but not of 1 and β.

Defining

$$\psi = \exp\left(\frac{i}{\hbar}\Lambda\right)\Psi, \quad \Lambda(x) = \int^x dx' \frac{V_e(x')}{c} \tag{3}$$

the space component of the vector potential is gauged away

$$\left(p + \frac{V_e}{c}\right)\Psi = \exp\left(\frac{i}{\hbar}\Lambda\right)p\psi \tag{4}$$

so that the time-independent Dirac equation can be rewritten as follows:

$$H\psi = E\psi, \quad H = \sigma_1 cp + \sigma_2 V_p + \sigma_3 \left(mc^2 + V_s\right) + 1V_t \tag{5}$$

showing that the space component of a vector potential only contributes to change the spinors by a local phase factor.

Provided that the spinor is written in terms of the upper and the lower components

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{6}$$

the Dirac equation decomposes into:

$$\left(V_t - E \pm V_s \pm mc^2\right) \psi_\pm = i\hbar c \psi'_\mp \pm i V_p \psi_\mp \tag{7}$$

where the prime denotes differentiation with respect to x. In terms of ψ_+ and ψ_- the spinor is normalized as $\int_{-\infty}^{+\infty} dx \left(|\psi_+|^2 + |\psi_-|^2\right) = 1$ so that ψ_+ and ψ_- are square integrable functions. It is clear from the pair of coupled first-order differential equations (7) that both ψ_+ and ψ_- have definite and opposite parities if the Dirac equation is covariant under $x \to -x$.

In the nonrelativistic approximation (potential energies small compared to mc^2 and $E \approx mc^2$) Eq. (7) becomes

$$\psi_- = \frac{p}{2mc}\,\psi_+ \tag{8}$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_t + V_s + \frac{V_p^2}{2mc^2} + \frac{\hbar V_p'}{2mc}\right)\psi_+ = \left(E - mc^2\right)\psi_+ \tag{9}$$

Eq. (8) shows that ψ_- is of order $v/c << 1$ relative to ψ_+ and Eq. (9) shows that ψ_+ obeys the Schrödinger equation without distinguishing the contributions of vector and scalar potentials (at this point the author digress to make his apologies for mentioning in former papers ([29]-[30]) that the pseudoscalar potential does not present any contributions in the nonrelativistic limit).

It is remarkable that the Dirac equation with a nonvector potential, or a vector potential contaminated with some scalar or pseudoscalar coupling, is not invariant under $V \to V+$const, this is so because only the vector potential couples to the positive-energies in the same way it couples to the negative-ones, whereas nonvector contaminants couple to the mass of the fermion. Therefore, if there is any nonvector coupling the absolute values of the energy

will have physical significance and the freedom to choose a zero-energy will be lost. This last statement remains truthfully in the nonrelativistic limit if one considers that such a contaminant is a pseudoscalar potential. It is also noticeable that the pseudoscalar coupling results in the Schrödinger equation with an effective potential in the nonrelativistic limit, and not with the original potential itself. Indeed, this is the side effect which in a 3+1 dimensional space-time makes the linear potential to manifest itself as a harmonic oscillator plus a strong spin-orbit coupling in the nonrelativistic limit [31]. The form in which the original potential appears in the effective potential, the V_p^2 term, allows us to infer that even a potential unbounded from below could be a confining potential. This phenomenon is inconceivable if one starts with the original potential in the nonrelativistic equation.

As we will see explicitly in the next section, a constant added to the pseudoscalar Coulomb potential is undoubtedly physically relevant. As a matter of fact, it plays a crucial role to ensure the existence of bounded solutions. Nevertheless, the resulting potential does not present any nonrelativistic limit. Oddly enough, $E \approx mc^2$ only for the positive-ground-state solution in the strong-coupling regime of the theory.

3 The pseudoscalar screened Coulomb potential

From now on we shall restrict our discussion to a pure pseudoscalar potential. For $E \neq \pm mc^2$, the coupling between the upper and the lower components of the Dirac spinor can be formally eliminated when Eq. (7) is written as second-order differential equations:

$$-\frac{\hbar^2}{2m}\,\psi_\pm'' + V_{eff}^{[\pm]}\,\psi_\pm = E_{eff}\,\psi_\pm \tag{10}$$

where

$$E_{eff} = \frac{E^2 - m^2c^4}{2mc^2} \tag{11}$$

$$V_{eff}^{[\pm]} = \frac{V^2}{2mc^2} \pm \frac{\hbar}{2mc}V' \tag{12}$$

These last results show that the solution for this class of problem consists

in searching for bounded solutions for two Schrödinger equations. It should not be forgotten, though, that the equations for ψ_+ or ψ_- are not indeed independent because the effective eigenvalue, E_{eff}, appears in both equations. Therefore, one has to search for bound-state solutions for $V_{eff}^{[+]}$ and $V_{eff}^{[-]}$ with a common eigenvalue. The Dirac eigenvalues are obtained by inserting the effective eigenvalues in (11).

The solutions for $E=\pm mc^2$, excluded from the Sturm-Liouville problem, can be obtained directly from the Dirac equation (7). One can observe that such a sort of isolated solutions can be written as

$$\begin{aligned} \psi_{\mp} &= N_{\mp}\, \exp\left[\mp v(x)\right] \\ \psi_{\pm}' \mp v'\psi_{\pm} &= \pm i\,\frac{2mc}{\hbar} N_{\mp}\, \exp\left[\mp v(x)\right] \end{aligned} \tag{13}$$

where N_+ and N_- are normalization constants and $v(x)=\int^x dy\, V(y)\,/(\hbar c)$. One can check that it is impossible to have both components different from zero simultaneously on the same side of the x-axis. Of course normalizable eigenstates are possible only if $v(x)$ has a distinctive leading asymptotic behaviour.

Now let us focus our attention on a pseudoscalar screened Coulomb potential in the form

$$V=\frac{\hbar c}{2\lambda}\left[g_0 - g\exp\left(-\frac{|x|}{\lambda}\right)\right] \tag{14}$$

where the coupling constants, g_0 and g, are real numbers and λ is a positive parameter related to the range of the interaction. Although $g<0$ gives rise to an ubiquitous repulsive potential in a nonrelativistic theory, the possibility of such a sort of potential to bind fermions is already noticeable in the nonrelativistic limit of the Dirac theory (see Eq. (9)). The presence of the uniform background potential, $\hbar c g_0/(2\lambda)$, is a distinguishing feature from the screened Coulomb potential in Refs. [25]-[28]. As one will see, it can not vanish for a binding pseudoscalar potential, and there is a minimum size for the coupling constant g_0 before any bounded solution can be obtained, viz. $|g_0|=1$. Evidently, the bound-state solutions for this potential present no nonrelativistic limit, hence we have to cope with an intrinsically relativistic bound-state problem.

We begin with the isolated solutions. In this case

$$v(x)=\frac{g_0}{2\lambda}\,x+\varepsilon(x)\frac{g}{2}\exp\left(-\frac{|x|}{\lambda}\right) \tag{15}$$

where $\varepsilon(x)$ stands for the sign function ($\varepsilon(x)=x/|x|$ for $x\neq 0$). It is not difficult to see that ψ_+ and ψ_- are not continuous at $x=0$, and for that reason those solutions should be discarded.

As for $E\neq 0$, the effective potential becomes the Morse-like potential

$$V_{eff}^{[\pm]}=A\exp\left(-2\frac{|x|}{\lambda}\right)+B_{\pm}\exp\left(-\frac{|x|}{\lambda}\right)+C \tag{16}$$

where

$$A=\frac{\hbar^2 g^2}{8\lambda^2 m}>0,\quad B_{\pm}=-\frac{\hbar^2 g}{4\lambda^2 m}\left[g_0\mp\varepsilon(x)\right],\quad C=\frac{\hbar^2 g_0^2}{8\lambda^2 m}>0 \tag{17}$$

Before proceeding, it is useful to make some qualitative arguments regarding the Morse-like potential and its possible solutions. The effective potential is able to bind fermions on the condition that $B_{\pm}<0$, contrariwise $V_{eff}^{[+]}$, or $V_{eff}^{[-]}$, or both of them would be repulsive everywhere. It follows that $\varepsilon(g)g_0>1$. It also follows that $V_{\min}^{[\pm]}<E_{eff}<C$, where $V_{\min}^{[\pm]}$ is the lowest value of $V_{eff}^{[\pm]}$. This implies that the Dirac eigenvalues corresponding to bounded solutions are in the range $m^2c^4<E^2<m^2c^4+2mc^2C$. There is a spectral gap in the range $E^2<m^2c^4$, and the energies belonging to $E^2>m^2c^4+2mc^2C$ correspond to the continuum.

Note that the range of discrete Dirac eigenvalues enlarges as λ decreases or $|g_0|$ increases. As a matter of fact, when λ decreases the potential well becomes deeper and beyond this it becomes narrower in such a way that $\lambda^2 V_{\min}^{[\pm]}$ is approximately constant, thus the capacity of the potential well to hold bound-state solutions remains almost the same. As for $|g_0|$, its increasing makes the potential well deeper almost without modifying its width. In this way, one expects that the parameter g_0 determinates the number of allowed bounded solutions.

The Morse-like potential presents, for $B_{\pm}<0$, a structure of asymmetric potential wells and the highest well governs the value of the zero-point energy. When $g\to 0$, $V_{eff}^{[\pm]}$ has a single-well structure and nondegenerate Dirac eigenvalues are expected. When $g\to\infty$, though, there appears a

double-well structure with an infinitely high and broad potential wall at the origin separating the two wells. In this last circumstance one would expect doubly-degenerate energy levels.

Note that the parameters of the effective potential are related in such a manner that the change $x \rightarrow -x$ induces the change $V_{eff}^{[\pm]} \rightarrow V_{eff}^{[\mp]}$ without affecting the effective energy, signifying that $|\psi_{\pm}(-x)|$ behaves like $|\psi_{\mp}(x)|$ and permitting us to focus our attention for a while on the positive side of the x-axis.

Now let us do it quantitatively. Defining the dimensionless quantities z, $\mu_{\varepsilon(g)}^{[\pm]}$ and ν,

$$z = |g| \exp\left(-\frac{x}{\lambda}\right)$$

$$\mu_{\varepsilon(g)}^{[\pm]} = \frac{\varepsilon(g)}{2}\,(g_0 \mp 1) \tag{18}$$

$$\nu = \frac{\lambda}{\hbar c}\sqrt{\left(\frac{\hbar c g_0}{2\lambda}\right)^2 + m^2c^4 - E^2}$$

and using (10)-(11) and (16)-(17), one obtains the equation on the positive side of the x-axis

$$z\,\psi_{\pm}'' + \psi_{\pm}' + \left(-\frac{z}{4} - \frac{\nu^2}{z} + \mu_{\varepsilon(g)}^{[\pm]}\right)\psi_{\pm} = 0 \tag{19}$$

Now the prime denotes differentiation with respect to z. Following the steps of Refs. [27] and [28], we make the transformation $\psi_{\pm} = z^{-1/2}\Phi_{\pm}$ to obtain the Whittaker equation [48]:

$$\Phi_{\pm}'' + \left(-\frac{1}{4} + \frac{\mu_{\varepsilon(g)}^{[\pm]}}{z} + \frac{1/4 - \nu^2}{z^2}\right)\Phi_{\pm} = 0 \tag{20}$$

whose solution vanishing at the infinity becomes

$$\Phi_{\pm} = N_{\pm}\, z^{\nu+1/2} e^{-z/2} M\left(a_{\varepsilon(g)}^{[\pm]}, b, z\right) \tag{21}$$

where N_{+} and N_{-} are constants, and M is the regular solution of the confluent hypergeometric equation (Kummer´s equation) [48]:

$$zM'' + (b-z)M' - a^{[\pm]}_{\varepsilon(g)} M = 0 \tag{22}$$

with

$$a^{[\pm]}_{\varepsilon(g)} = \nu + \frac{1}{2} - \frac{\varepsilon(g)}{2}\left(g_0 \mp 1\right), \qquad b = 2\nu + 1 \tag{23}$$

Now we are ready to write the physically acceptable solutions on both sides of the x-axis by recurring to the symmetry $|\psi_\pm(-x)| \sim |\psi_\mp(x)|$ mentioned before. They are

$$\begin{aligned}
\psi_+ &= z^\nu e^{-z/2}\left[\theta(-x) C^{[-]} M\left(a^{[-]}_{\varepsilon(g)}, b, z\right) + \theta(+x) C^{[+]} M\left(a^{[+]}_{\varepsilon(g)}, b, z\right)\right] \\
\psi_- &= z^\mu e^{-z/2}\left[\theta(-x) D^{[-]} M\left(a^{[+]}_{\varepsilon(g)}, b, z\right) + \theta(+x) D^{[+]} M\left(a^{[-]}_{\varepsilon(g)}, b, z\right)\right]
\end{aligned} \tag{24}$$

where $C^{[\pm]}$ and $D^{[\pm]}$ are constants, and $\theta(x)$ is the Heaviside function.

The continuity of the wavefunctions (24) at $x = 0$ furnishes

$$C^{[+]} D^{[+]} = C^{[-]} D^{[-]} \tag{25}$$

$$\frac{C^{[+]}}{D^{(+)}} = \frac{C^{[-]}}{D^{[-]}}\left[\frac{M\left(a^{[+]}_{\varepsilon(g)} - \varepsilon(g), b, z_0\right)}{M\left(a^{[+]}_{\varepsilon(g)}, b, z_0\right)}\right]^2 \tag{26}$$

where $z_0 = |g|$. Substituting the solutions (24) into the Dirac equation (7) and making use of the recurrence formulas [48]

$$\begin{gathered}
M'(a,b,z) = \frac{a}{b} M(a+1, b+1, z) \\
bM(a,b,z) - bM(a-1,b,z) - zM(a,b+1,z) = 0
\end{gathered} \tag{27}$$

one has as a result

$$\frac{C^{[\pm]}}{D^{[\pm]}} = \pm\left[\frac{i\hbar c}{\lambda}\,\frac{a^{[-\varepsilon(g)]}_{\varepsilon(g)}}{E \mp \varepsilon(g) mc^2}\right]^{\pm\varepsilon(g)} \tag{28}$$

Together, (26) and (28) lead to the quantization condition

$$\left[\frac{M\left(a^{[+]}_{\varepsilon(g)} - \varepsilon(g), b, z_0\right)}{M\left(a^{[+]}_{\varepsilon(g)}, b, z_0\right)} \right]^2 + 1 - \frac{g_0}{a^{[+]}_{-}} = 0 \tag{29}$$

This last equation only holds for $g_0/a^{[+]}_{-} > 1$, and it represents a limit imposed on E^2; namely, $E^2 > m^2c^4$, as has been anticipated by the preceding qualitative arguments. The transcendental eigenvalue equation (29) is invariant under the transformation $E \to -E$ as well as under the combined transformations $g_0 \to -g_0$ and $g \to -g$, and it can be solved easily with a symbolic algebra program by searching for solutions in the range $m^2c^4 < E^2 < m^2c^4 + (\hbar c g_0/2\lambda)^2$, with $\varepsilon(g)g_0 > 1$. Fig. 1 shows the behaviour of the positive-eigenenergies as a function of g for λ equal to the Compton wavelength, i.e. $\lambda = \hbar/(mc)$, and $g_0 = 5$. There is a finite number of eigenvalues. Notice that for very small values of g, at least one energy level emerging from the continuum comes into existence. The ground state tends asymptotically to $\pm mc^2$ for arbitrary large g whereas the excited states tend to be closely bunched in pairs with eigenvalues independent of g. Due to the appearance of the potential barrier for large g, this sort of two-fold degeneracy comes as no surprise.

Now we return our attention to (24), where it remains to evaluate the constants $C^{[\pm]}$ and $D^{[\pm]}$. Using (25) and (28) one obtains

$$D^{[+]} = \left[\frac{i\hbar c}{\lambda} \frac{a^{[\varepsilon(g)]}_{-\varepsilon(g)}}{E + \varepsilon(g)mc^2} \right]^{\varepsilon(g)} C^{[+]}$$

$$C^{[-]} = \left[\frac{\hbar c}{\lambda} \frac{a^{[\varepsilon(g)]}_{-\varepsilon(g)}}{\sqrt{E^2 - m^2c^4}} \right]^{\varepsilon(g)} C^{[+]} \tag{30}$$

$$D^{[-]} = i\sqrt{\frac{E - mc^2}{E + mc^2}}\, C^{[+]}$$

and the constant $C^{[+]}$ is to be fixed by the normalization condition. It is noticeable in (24) a somewhat left-right symmetry involving ψ_+ and ψ_-, such a symmetry is not exact inasmuch as $C^{[\pm]} \neq D^{[\mp]}$. The upper and lower components of the spinor have the same number of zeros, or nodes, and as a consequence of such a quasi-symmetry the zeros of ψ_+ and ψ_- exhibit,

of course, an exact left-right symmetry. In whatever manner Eq. (30) of course, is invariant under the change $E \to -E$ provided $|C^{[\pm]}| \leftrightarrow |D^{[\mp]}|$, hence one can conclude that $|\psi_+(\pm x)| \leftrightarrow |\psi_-(\mp x)|$ in such a way that the position probability density on the right side of the x-axis transforms into the position probability density on the left side of the x-axis, and vice versa. As for the combined transformations $g_0 \to -g_0$ and $g \to -g$, one has $|C^{[\pm]}| \leftrightarrow |D^{[\pm]}|$ with the proviso that $m \to -m$. Recalling that the existence for bounded solutions demands that $\varepsilon(g)g_0 > 1$, this last set of mathematical transformations allow us to concentrate our attention on the case $g > 0$. Figs. 2-4 illustrate the behaviour of the upper and lower components of the Dirac spinor, $|\psi_+|^2$ and $|\psi_-|^2$, and the position probability density, $|\psi|^2 = |\psi_+|^2 + |\psi_-|^2$, as a function of x, for the positive-energy solutions with $g = 2$, $g_0 = 5$ and $\lambda = \hbar/(mc)$. Figs. 5-7 do the same with $g = 14$. The normalization constant, $C^{[+]}$, was fixed by numerical computation. Comparison of these figures shows that $|\psi_+|$ is larger than $|\psi_-|$ (for $E > 0$) and that the fermion tends to avoid the origin as g increases. Furthermore, $|\psi|$ tends to concentrate at the left (right) region when E and g have equal (different) signs. This happens due to the discontinuity of the effective Morse-like potential at $x = 0$, given by $V_{eff}^{[\pm]}(0_+) - V_{eff}^{[\pm]}(0_-) = \pm\, \hbar^2 g/(2\lambda^2 m)$. Note from all these figures that the quantum number n qualifies the number of nodes of ψ_+ and ψ_-. Note also that the quasi-degeneracy of the second- and third-excited states for $g = 14$ reveals a near equality of the position probability densities for those states, as should be expected.

A numerical calculation of the uncertainty in the position for the ground-state solution (with $m = c = \hbar = 1$ and $g_0 = 5$) furnishes 0.736 and 0.469 for $g = 2$ and 14, respectively. Here we have purposely shown an odd fact for $g = 14$. It seems that the uncertainty principle dies away provided such a principle implies that it is impossible to localize a particle into a region of space less than half of its Compton wavelength (see, e.g., Ref. [49]). This apparent contradiction can be remedied by recurring to the concepts of effective mass and effective Compton wavelength. Indeed, the third line of (18) suggests that we can define the effective mass as

$$m_{eff} = \sqrt{m^2 + \left(\frac{\hbar g_0}{2\lambda c}\right)^2} \tag{31}$$

Hence, the effective Compton wavelength can be defined as $\lambda_{eff} = \hbar/(m_{eff}c)$ so that the minimum uncertainty consonant with the uncertainty principle

is given by $\lambda_{eff}/2$. Therefore, the uncertainty in the position can shrink without limit as $|g_0|$ increases or λ decreases. It means that the localization of the fermion does not require any minimum value as $|g_0| \rightarrow \infty$ or $\lambda \rightarrow 0$ in order to ensure the single-particle interpretation of the Dirac equation.

4 Conclusions

We have succeed in searching for Dirac bounded solutions for neutral fermions by considering a pseudoscalar screened Coulomb potential in 1+1 dimensions. The satisfactory completion of this task has been alleviated by the methodology of effective potentials which has transmuted the question for $E \neq \pm mc^2$ into Schrödinger-like equations with effective Morse-like potentials. There are no isolated solutions ($E = \pm mc^2$) for this sort of potential. Nevertheless, as the coupling constant g becomes extremely large, i.e. $|g| \gg 1$, the energy levels corresponding to the ground-state solution end up close to $\pm mc^2$.

The uniform background potential, beyond the screened Coulomb potential, is a sine qua non condition for the existence of bound-state solutions. Curiously enough, it plays an essential role not only to determinate the attractiveness of the effective Morse-like potential but also for establishing the band of allowed discrete Dirac eigenvalues. Furthermore, due to the fact that there is no atmosphere for the production of fermion-antifermion pairs, a neutron fermion embedded in this uniform background acquires an effective mass which permits that it can be strictly localized.

In addition to their intrinsic importance as new solution of a fundamental equation in physics, the solutions obtained in this paper might be of relevance to the confinement of neutral fermions in a four-dimensional world. Furthermore, they render a contrast to the result and conclusions found in [40]: there are bound-state solutions for neutral fermions interacting by a pseudoscalar screened Coulomb potential in 1+1 dimensions, notwithstanding the spinor is not an eigenfunction of the parity operator.

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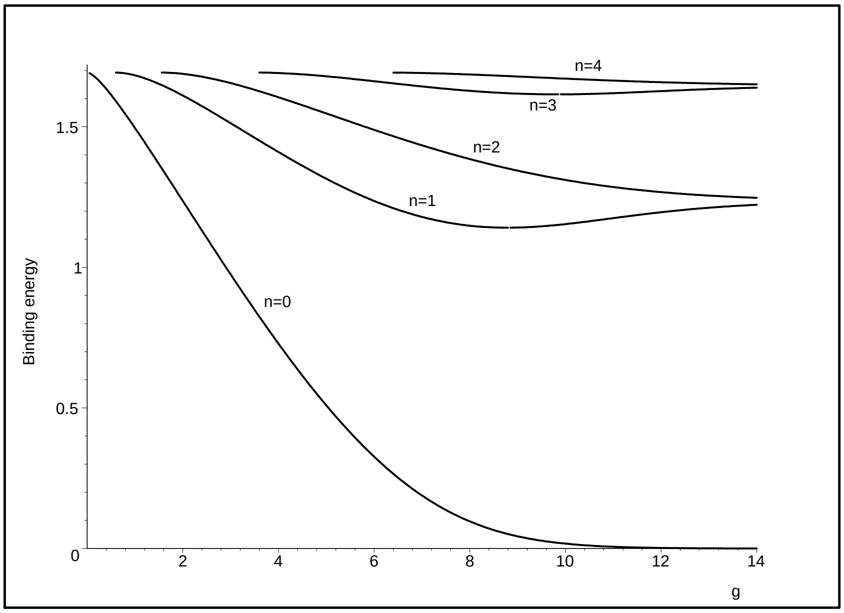

Figure 1: Binding energies ($E - mc^2$) corresponding to positive Dirac eigenvalues as a function of g with $\lambda = \hbar/(mc)$ and $g_0 = 5$ ($m = c = \hbar = 1$).

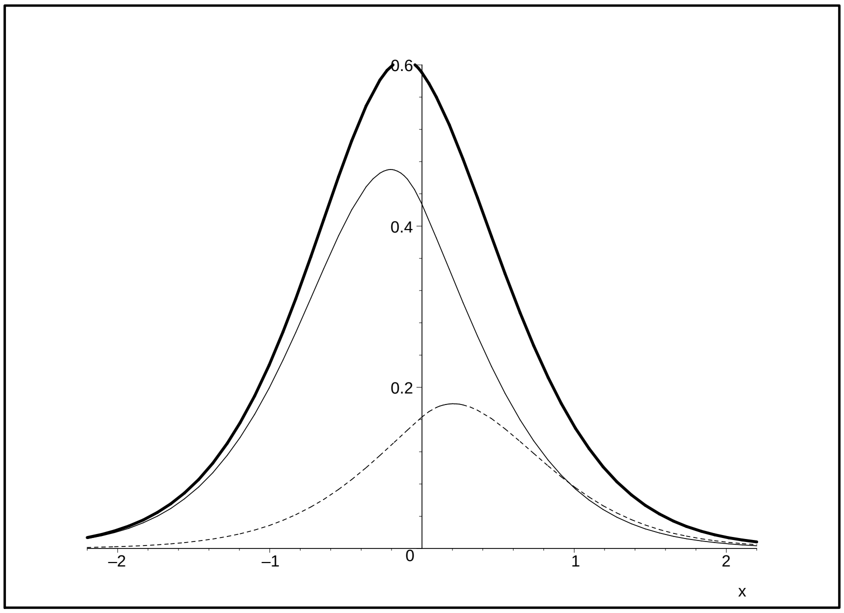

Figure 2: $|\psi_+|^2$ (full thin line), $|\psi_-|^2$ (dashed line) and $|\psi|^2 = |\psi_+|^2 + |\psi_-|^2$ (full thick line) as a function of x, corresponding to the positive-ground-state energy ($n = 0$) with $\lambda = \hbar/(mc)$, $g = 2$ and $g_0 = 5$ ($m = c = \hbar = 1$).

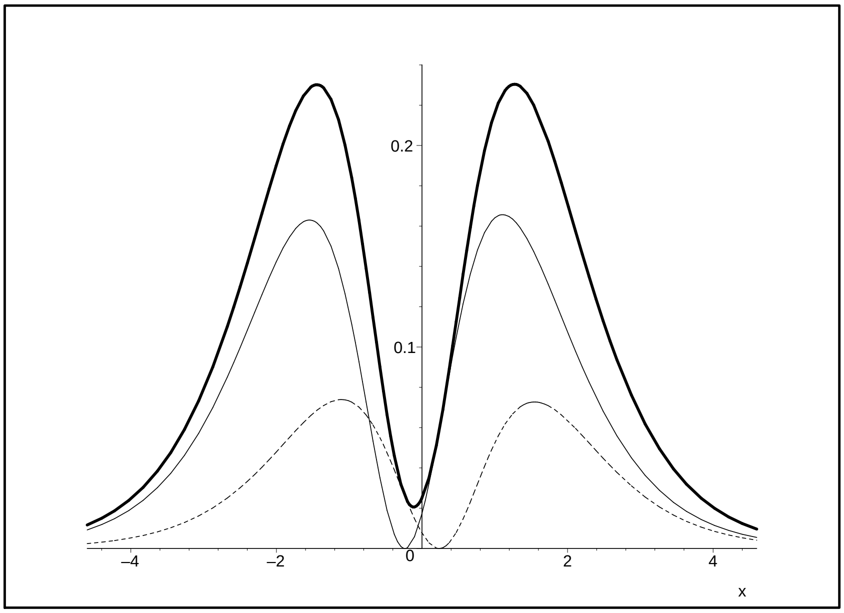

Figure 3: The same as in Fig. 2 corresponding to positive-first-excited-state energy $(n = 1)$.

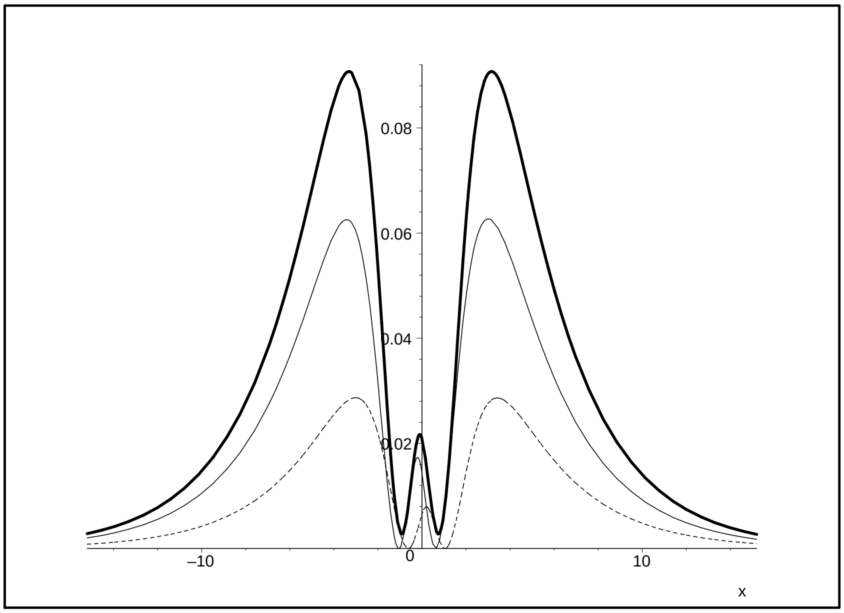

Figure 4: The same as in Fig. 2 corresponding to positive-second-excited-state energy ($n = 2$).

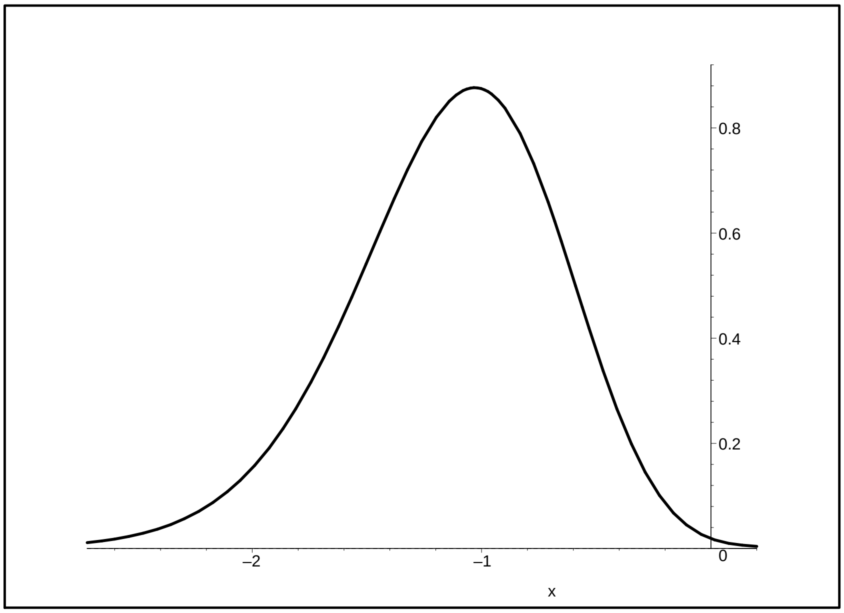

Figure 5: The same as in Fig. 2 for $g = 14$. Here $|\psi_-|^2 \ll |\psi_+|^2$, hence $|\psi|^2 \approx |\psi_+|^2$.

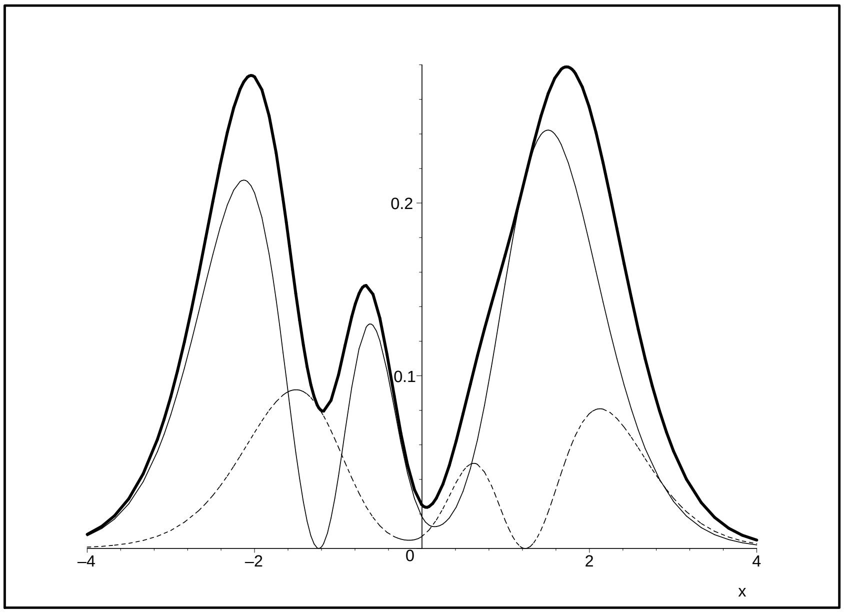

Figure 6: The same as in Fig. 2 for $n = 1$ and $g = 14$.

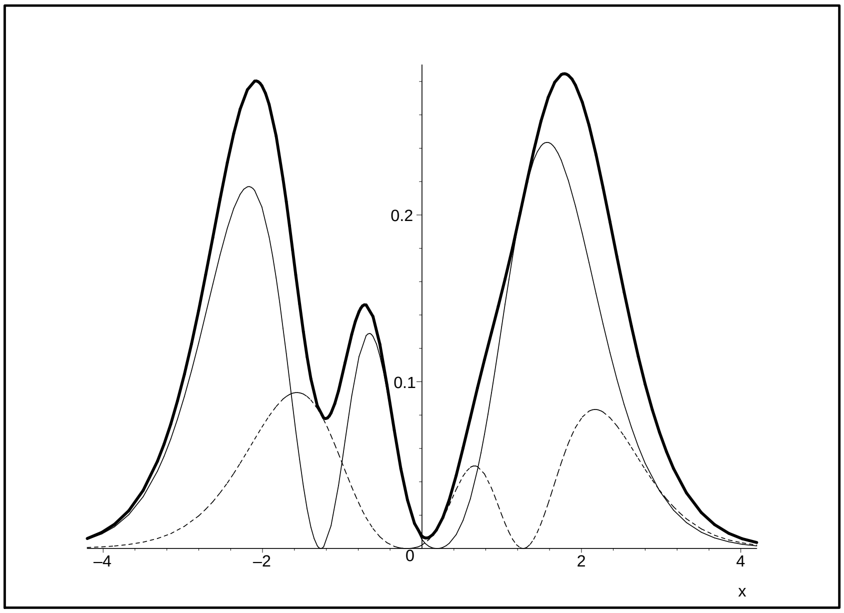

Figure 7: The same as in Fig. 2 for $n = 2$ and $g = 14$.